FARTHEST POINTS OF SETS IN UNIFORMLY CONVEX BANACH SPACES

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ABSTRACT
Let S be a closed and bounded set in a uniformly convex Banach space X. It is shown that the set of all points in X which have a farthest point in S is dense. Let $b(S)$ denote the set of all farthest points of S, then a sufficient condition for $\overline{\text{co}}\, S = \overline{\text{co}}\, b(S)$ to hold is that X have the following property (I): Every closed and bounded convex set is the intersection of a family of closed balls.

1. Let S be a subset of a normed linear space and let $b(S)$ denote the set of all $s \in S$ for which an element c exists such that

$$(*) \qquad \| s - c \| = \sup \{ \| x - c \| \mid x \in S \}$$

i.e. the set of all farthest points in S. In [3] we proved that if S is a closed and bounded set in Hilbert space then $b(S) \neq \emptyset$. Asplund [1] proved independently that in the case of a convex closed and bounded S in a Hilbert space H, S is identical with the closed convex hull, $\overline{\text{co}}\, b(S)$, of the set of farthest points; in addition he showed [2]* that the set C of all points in H satisfying (*) for some $s \in S$ is dense (in H). In the present note we show that the last result is true for any closed and bounded set in a uniformly convex Banach space X. If, in addition X has a certain smoothness property (I) (cf. section 3), known to hold for all reflexive spaces having a strongly differentiable norm, then $\overline{\text{co}}\, S = \overline{\text{co}}\, b(S)$.

2. In a normed linear space X let $V = \{x \mid \| x \| \leq 1\}$. For any real ε, $0 < \varepsilon \leq 2$, define the function $\delta(\varepsilon)$, called the modulus of convexity of X, by setting

$$(1) \qquad \delta(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \| x + y \| \;\middle|\; x, y \in V, \| x - y \| \geq \varepsilon \right\}$$

The space X is called uniformly convex if $\delta(\varepsilon) > 0$ for all ε in the domain of definition of δ. Clearly

$$(2) \qquad \varepsilon \geq \varepsilon' \Rightarrow \delta(\varepsilon) \geq \delta(\varepsilon')$$

Also, as is readily verified,

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$$\delta(\varepsilon) \leqq \frac{\varepsilon}{2} \tag{3}$$

LEMMA 1. *Let $x, y \in V$, $x \neq y$, and suppose $0 < \mu < \frac{1}{2}$; then*

$$1 - \| \mu x + (1-\mu)y \| \geqq 2\mu\delta(\| x - y \|) \tag{4}$$

Proof. Let $z = \mu x + (1-\mu)y$. It clearly suffices to show that all $w \in X$ with $\| w - z \| \leqq 2\mu\delta(\| x - y \|)$ are in V. Set $v = \frac{1}{2}\mu(w - (1-2\mu)y)$. Then $w = 2\mu v + (1-2\mu)y$ is a convex combination of v and y and it suffices to show that v is within distance $\delta(\| x - y \|)$ from $\frac{1}{2}(x+y)$.
Now

$$\begin{aligned} \left\| v - \frac{1}{2}(x+y) \right\| &= \frac{1}{2\mu} \| w - (1-2\mu)y - \mu(x+y) \| \\ &= \frac{1}{2\mu} \| w - \mu x - (1-\mu)y \| \\ &= \frac{1}{2\mu} \| w - z \| \leqq \delta(\| x - y \|). \end{aligned}$$

LEMMA 2. *Let $0 < \alpha < 1$, $0 < \beta < \frac{1}{2}$ and suppose $x, y \in X$ and $f \in X^*$ satisfy the following conditions*

$$\| x \| \leqq 1 = \| y \| = f(y) = \| f \| \tag{5}$$

$$f(x) \leqq 1 - \alpha \tag{6}$$

$$\| x - \beta y \| \leqq 1 - \beta \tag{7}$$

Then

$$\| x \| \leqq 1 - 2\beta\delta(\alpha) \tag{8}$$

Proof. Let $u = (1/(1-\beta))(x - \beta y)$; then $\| u \| \leqq 1$ and $x = \beta y + (1-\beta)u$. It follows from Lemma 1 that $1 - \| x \| \geqq 2\beta\delta(\| u - y \|) \geqq 2\beta\delta(\| x - y \|)$. Now $\| y - x \| \geqq \| f \| \; \| y - x \| \geqq f(y - x) = f(y) - f(x) \geqq \alpha$. Thus $\| x \| \leqq 1 - 2\beta\delta(\alpha)$ as asserted.

THEOREM 1. *Let S be a nonempty closed and bounded set in a uniformly convex Banach space X. Then the set C, of all points c in X for which there is a point $s \in S$ with $\| s - c \| = \sup\{\| x - c \| \mid x \in S\}$, is dense (in X).*

Proof. Given $c_0 \in X$ let

$$r_1 = \sup\{\| x - c_0 \| \mid x \in S\} \tag{9_1}$$

We may clearly assume that $r_1 > 0$. To prove the theorem it suffices to show that for an arbitrary p, $0 < p < r_1$, there is a $c \in X$, as required, with $\| c - c_0 \| \leqq p$.

To this end we define inductively sequences $\{c_n\}$ and $\{x_n\}$, $n = 1,2,\cdots$, converging to c and s respectively. Let, then, $x_1 \in S$ be chosen so that

$$(10_1) \qquad \| x_1 - c_0 \| \geqq r_1 \left(1 - \frac{p}{2r_1}\, \delta^2(1)\right)$$

Next, let

$$(11_1) \qquad c_1 = c_0 + \frac{c_0 - x_1}{\| c_0 - x_1 \|}\, \frac{p}{2}$$

Assuming r_{n-1}, x_{n-1} and c_{n-1} already defined set

$$(9_n) \qquad r_n = \sup\{\| x - c_{n-1} \| \quad x \in S\}$$

and choose $x_n \in S$ so that

$$(10_n) \qquad \| x_n - c_{n-1} \| \geqq r_n \left(1 - \frac{p}{2^n r_n}\, \delta^{n+1}(1)\right).$$

Finally, let

$$(11_n) \qquad c_n = c_{n-1} + \frac{c_{n-1} - x_n}{\| c_{n-1} - x_n \|}\, \frac{p}{2^n}.$$

Of the sequences $\{r_n\}$, $\{x_n\}$ and $\{c_n\}$ thus defined the last one is clearly a Cauchy sequence by (11_n). We proceed to show that so is $\{x_n\}$. For each positive integer n let then $R_n > 0$, $\frac{1}{2} > \beta_n > 0$ and $f_n \in X^*$ be defined as follows.

$$(12_n) \qquad R_n = r_n + 2^{-n} p$$

$$(13_n) \qquad \beta_n = 2^{-n}\frac{p}{R_n}, \quad u_n = \frac{x_n - c_{n-1}}{\| x_n - c_{n-1} \|}$$

and

$$(14_n) \qquad f_n(u_n) = \| f_n \| = 1$$

From (9_{n+1}), (11_n), (10_n) and (12_n) we get

$$(15_n) \qquad r_{n+1} \geqq R_n - \frac{p}{2^n}\, \delta^{n+1}(1)$$

It follows from (9_n), (11_n) and (12_n) that both x_n and x_{n+1} satisfy the inequality

$$(16_n) \qquad \left\| \frac{z - c_n}{R_n} \right\| \leqq 1$$

Further,

$$
\begin{aligned}
f_n(x_n-c_n) &= f_n(x_n-c_{n-1})+f_n(c_{n-1}-c_n)=\|x_n-c_{n-1}\|+2^{-n}p \\
&\geqq r_n-\frac{p}{2^n}\,\delta^{n+1}(1)+2^{-n}p=R_n\left(1-\frac{p}{2^nR_n}\,\delta^{n+1}(1)\right) \\
&> R_n(1-\delta^n(1))
\end{aligned} \tag{17_n}
$$

To complete the proof that $\{x_n\}$ is a Cauchy sequence it suffices now to show that

$$f_n(x_{n+1}-c_n)>R_n(1-\delta^n(1)) \tag{18_n}$$

Indeed (16_n), (17_n) and (18_n) are easily seen to imply

$$\|x_{n+1}-x_n\|\leqq R_n\delta^{n-1}(1)\leqq(r_1+p)\delta^{n-1}(1)\leqq 2^{-n+1}(r_1+p)\,(*)$$

To establish (18_n) we make use of Lemma 2.
We note, then, that

$$\left\|\frac{x_{n+1}-c_n}{R_n}-\beta_n u_n\right\|\leqq 1-\beta_n$$

$$\Big[\text{For } \|x_{n+1}-c_{n-1}\|\leqq r_n=R_n-2^{-n}p=R_n(1-\beta_n)$$

and

$$\frac{x_{n+1}-c_{n-1}}{R_n}=\frac{x_{n+1}-c_n}{R_n}+\frac{c_n-c_{n-1}}{R_n}=\frac{x_{n+1}-c_n}{R_n}-\beta_nu_n.\Big]$$

and using (10_{n+1}) and (15_n)

$$
\begin{aligned}
\|x_{n+1}-c_n\| &\geqq r_{n+1}\left(1-\frac{p}{2^{n+1}r_{n+1}}\,\delta^{n+2}(1)\right) \\
&\geqq R_n-\frac{p}{2^n}\,\delta^{n+1}(1)-\frac{p}{2^{n+1}}\,\delta^{n+2}(1) \\
&> R_n-\frac{p}{2^{n-1}}\,\delta^{n+1}(1)=R_n(1-2\beta_n\delta^{n+1}(1))
\end{aligned}
$$

(*) For

$$
\begin{aligned}
\frac{1}{2}\left\|\frac{x_{n+1}-c_n}{R_n}+\frac{x_n-c_n}{R_n}\right\| &\geqq \frac{1}{2}\,f_n\left(\frac{x_{n+1}-c_n}{R_n}+\frac{x_n-c_n}{R_n}\right) \\
&> 1-\delta^n(1)
\end{aligned}
$$

and, it clearly follows from (1), that $\|x_{n+1}-x_n\|\leqq R_n\delta^{\;-1}(1)$.

Thus (setting $\alpha = \delta^{n+1}(1)$) we obtain

$$f_n(x_{n+1} - c_n) > R_n(1 - \delta^n(1))$$

as asserted.

Let now $s = \lim_{n\to\infty} x_n$ and suppose $c = \lim_{n\to\infty} c_n$.

We clearly have

$$\sup\{\| c - x \| \mid x \in S\} = \lim_{n\to\infty} (\sup\{\| c_n - x \| \mid x \in S\})$$

$$\lim_{n\to\infty} r_{n+1} = \lim_{n\to\infty} \| c_n - x_{n+1} \| = \| c - s \|$$

concluding the proof of the theorem.

REMARKS. In [5] Lindenstrauss defined the notion of a strongly exposed point as follows: A point $s \in S$ is said to be a strongly exposed point of S if there is an $f \in X^*$ such that $f(y) < f(s)$ for $y \neq s$ and whenever $\{x_n\} \subset S$ is such that $f(x_n) \to f(s)$ then $\| x_n - s \| \to 0$. Since every point on the boundary of the unit ball of a uniformly convex Banach space is known to be strongly exposed it follows from the above theorem that every closed and bounded set in a uniformly convex Banach space has strongly exposed points.

3. DEFINITION. A normed linear space X is said to have property (I) if every closed and bounded convex set in X can be represented as the intersection of a family of closed balls. This property was introduced by Mazur [6] and shown to hold for all reflexive Banach spaces having a strongly differentiable norm (cf. also Phelps [7, p. 976]).

THEOREM 2. *Let X and S be as in Theorem 1 and suppose, in addition, that X has property (I). Then*

$$\overline{\text{co}}\, S = \overline{\text{co}}\, b(S).$$

Proof. Clearly $\overline{\text{co}}\, b(S) \subset \overline{\text{co}}\, S$. To prove the reverse inclusion suppose $x \notin \overline{\text{co}}\, b(s)$. Then, by property (I) there is a closed ball

$$B(c_0, r) = \{y \mid \| y - c_0 \| \leqq r\},$$

where $c_0 \in X$ and $r > 0$, such that $\overline{\text{co}}\, b(S) \subset B(c_0, r)$ and $\| x - c_0 \| - r > 0$. By Theorem 1 there is a $c \in X$ such that $\| c - c_0 \| < \| x - c_0 \| - r$ with $c \in C$. If $s \in S$ is farthest from c then $\| s - c \| \leqq \| s - c_0 \| + \| c_0 - c \| < \| x - c_0 \|$ showing that $S \subset B(c_0, r)$. Thus $x \notin \overline{\text{co}}\, S$ and $\overline{\text{co}}\, S \subset \overline{\text{co}}\, b(S)$ completing the proof.

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